Math 132: Differential Topology

§ Transversality

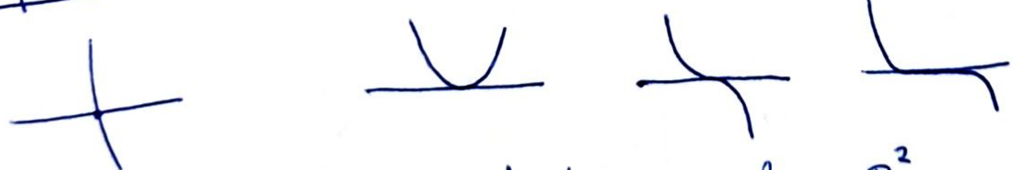
Def We say two smooth submanifolds $M, M' \subset N$ are transversal at $p \in M \cap M'$ if $T_pM + T_pM' = T_pN$.

If they are transversal at every point $p \in M \cap M'$, we simply say they are transversal, and write $M \pitchfork M'$.

Def More generally, we say a smooth map $f: M \to N$ is transversal to a submanifold $M' \subset N$ (abbreviated $f \pitchfork M'$) if $\text{Image}(df_p) + T_qM' = T_qN$ for every $p \in f^{-1}(M')$.

$q = f(p)$

Examples



transversal in $\mathbb{R}^2$ — not transversal in $\mathbb{R}^2$

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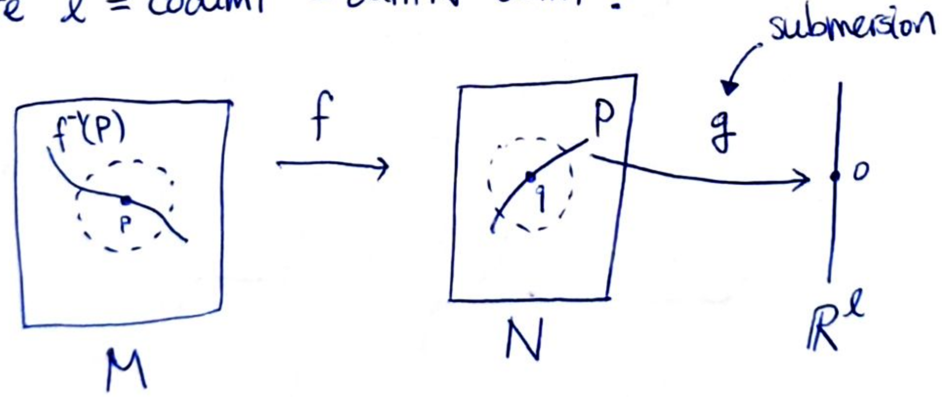
The following theorem generalizes the preimage theorem for regular values:

<u>Thm</u> If $f: M \to N$ is a smooth map transversal to a submanifold $P \subset N$, then the preimage $f^{-1}(P) \subset M$ is a submanifold of M, whose codimension (i.e. $\dim M - \dim f^{-1}(P)$) equals that of $P \subset N$ (i.e. $\dim N - \dim P$).

proof) We need to show that every point $p \in f^{-1}(P)$ has a neighborhood U in M such that $f^{-1}(P) \cap U$ is a manifold.

Let $q = f(p)$. In a neighborhood of q, we may write P as the zero set of a collection of independent functions $g_1, \cdots, g_\ell$, where $\ell = \operatorname{codim} P = \dim N - \dim P$.



Then, near p, $f^{-1}(P)$ is the zero set of $(g_1 \circ f, \cdots, g_\ell \circ f) = g \circ f$, so it's enough to show that 0 is a regular value of $g \circ f$.

Since $d(g\circ f)_p = dg_q \circ df_p$ and $dg_q : T_qN \to \mathbb{R}^\ell$ is surjective with $\ker dg_q = T_qP \subset T_qN$, 0 is a regular value of $g\circ f$ if and only if $\mathrm{Image}(df_p) + T_qP = T_qN$, which is exactly our transversality assumption. ▨

<u>Cor</u> If $M, M' \subset N$ are two transversal submanifolds, then $M\cap M'$ is also a submanifold, with

$$\operatorname{codim}(M\cap M') = \operatorname{codim} M + \operatorname{codim} M'.$$

<u>Rmk</u> Additivity of codimension is very natural.
Think: cutting out a submanifold locally using $k = \operatorname{codim} M$ and $\ell = \operatorname{codim} M'$ independent functions.

§ Homotopy and stability

<u>Def</u> We say two smooth maps $f_0, f_1 : M \to N$ are <u>homotopic</u>, abbreviated $f_0 \sim f_1$, if there exists a smooth map $F : M\times I \to N$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.
We call such F a <u>homotopy</u> between f_0 and f_1.

$f_t(x) = F(x,t)$ is a smoothly evolving family of maps

Homotopy is an equivalence relation on smooth maps from M to N, and the equivalence class to which a map belongs is called its <u>homotopy class</u>.

<u>Def</u> A property of smooth maps is called <u>stable</u> if whenever $f_0 : M \to N$ possesses the property and $f_t : M \to N$ is a homotopy of f_0, then, for some $\varepsilon > 0$, each f_t with $t < \varepsilon$ also possesses the property.

<u>Thm</u> (Stability theorem)

The following classes of smooth maps of a <u>compact</u> manifold M into a manifold N are stable classes:

- local diffeomorphisms
- immersions
- submersions
- maps transversal to a given closed submanifold $P \subset N$
- embeddings
- diffeomorphisms

(See [Guillemin-Pollack] pp. 36-37 for the proofs)